

Fierz-Pauli equation for massive gravitons from Induced Matter theory of gravity

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Abstract

Starting with a 5D physical vacuum described by a 5D Ricci-flat background metric, we study the emergence of gravitational waves (GW) from the Induced Matter (IM) theory of gravity. We obtain the equation of motion for GW on an 4D curved spacetime which has the form of a Fierz-Pauli one. In our model the mass of gravitons m_g is induced by a static foliation on the noncompact space-like extra dimension and the source-term is originated in the interaction of the GW with the induced connections of the background 5D metric. Here, relies the main difference of this formalism with the original Fierz-Pauli one.

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I. INTRODUCTION

Although standard general relativity (GR)[1] achieved great success and withstood many experimental tests, it also displayed many shortcomings and flaws which today make theoreticians question whether it is the definitive theory of gravity [see[2] and references therein]. It is well known that GR is very difficult to quantize. This fact rules out the possibility of treating gravitation like other quantum theories, and precludes the unification of gravity with other interactions. At the present time, it is not possible to realize a consistent quantum gravity theory which leads to the unification of gravitation with the other forces. In particular, the theory of gravitational waves (GW) is a rich subject that brings together different domains such as general relativity, field theory, astrophysics and cosmology. At present various gravitational-wave detectors, after decades of developments, have reached a sensitivity where there are significant chances of detection, and future improvements are expected to lead, in a few years, to advanced detectors with even better sensitivities[3]. The tensor perturbations of the metric $h_{\mu\nu}$ propagate and can affect the background space-time $\bar{g}_{\mu\nu}$ [4]. Since the discovery of the CMB electromagnetic radiation we know that its spectrum is a perfect black-body. This background is, to a first order approximation, isotropic. There are good reasons to expect that the Universe is permeated also by a stochastic background of GWs generated in the early universe. Furthermore, a stochastic background can also emerge from the incoherent superposition of a large number of astrophysical sources, too weak to be detected separately, and such that the number of sources that contribute to each frequency bin is much larger than one. The inflationary theory is one of the better candidates to describe the early stages of the accelerated expansion of the universe[5]. This theory can be recovered from the 5D IM one[6, 7].

In this letter we study the emergence of gravitational waves (GW) from the Induced Matter (IM) theory of gravity[8]. We shall investigate how we can obtain the equation that describes the evolution of GW from a 5D vacuum state, which is defined from a Ricci-flat spacetime. We shall restrict to canonical metrics[9] which are, at least, 5D Ricci-flat, on which we shall define a 5D vacuum state.

II. FORMALISM

We consider a 5D theory of gravity on which we define a vacuum, such that the first variation of the action is ${}^{(5)}\delta\mathcal{I} = {}^{(5)}\delta\mathcal{I}_E + {}^{(5)}\delta\mathcal{I}_M$, where

$${}^{(5)}\delta\mathcal{I} = \frac{1}{2} \int d^5x \sqrt{|g|} \delta g^{ab} \left\{ \frac{\mathcal{R}_{ab}}{8\pi G} + T_{ab} \right\}. \quad (1)$$

Here, the first term is the variation of the gravitational Einstein action ${}^{(5)}\mathcal{I}_E$ and the second one is the variation of the matter action ${}^{(5)}\mathcal{I}_M$. Here, G is the gravitational constant, g is the determinant of the covariant tensor metric $g_{ab} = \bar{g}_{ab} + \delta g_{ab}$ ¹ and $\mathcal{R} = \bar{\mathcal{R}} + \delta\mathcal{R}$ is the Ricci scalar on the metric. Furthermore, \bar{g}_{ab} ² is the background tensor metric, which we shall consider as describing a Ricci-flat spacetime. The energy-momentum tensor of matter, T_{ab} is defined from the variation of the matter action ${}^{(5)}\mathcal{I}_M$ under a change of the metric, and will be considered as null to describe the 5D apparent vacuum. In what follows we shall take in mind a particular class of background metrics named canonical[10]

$$d\bar{S}^2 = \frac{\psi^2}{\psi_0^2} d\bar{s}^2 - d\psi^2, \quad (2)$$

where $d\bar{s}^2 = \bar{g}_{\alpha\beta}(x^\alpha, \psi) dx^\alpha dx^\beta$ and ψ_0 is introduced to preserve the physical dimensions. The perturbations, which are in principle very small, are $\delta g_{ab} = h_{ab}$. Since the fluctuations of the metric are small, the perturbed Riemann tensor can be linearized

$$\delta\mathcal{R}_{abcd} \simeq -\frac{1}{2} [h_{bd,ac} + h_{ac,bd} - h_{bc,ad} - h_{ad,bc}], \quad (3)$$

so that the Ricci tensor (which is equal to the perturbed Ricci tensor because the background Ricci tensor will be considered as zero) $\delta\mathcal{R}_{ab} = \bar{g}^{cd}\delta\mathcal{R}_{cabd}$

$$\delta\mathcal{R}_{ab} \simeq -\frac{1}{2} [h^c_{a,cb} + h^c_{b,ac} - \bar{g}^{ce}h_{ce,ab} - \square h_{ab}], \quad (4)$$

where commas denote the partial derivative and \square is the D'Alembertian on the background 5D spacetime.

¹ In this letter a, b run from 0 to 4 and Greek letters run from 0 to 3.

² Greek indices run from 0 to 3, and arabic ones from 0 from 4.

A. 5D Gauge invariant field

Now we consider the gauge invariant field $\Psi_b^a = h_b^a - \frac{1}{2}\delta_b^a h$, where h is the scalar $h = \bar{g}^{ab}h_{ab}$, such that we impose the 5D gauge

$$\Psi_{b;a}^a = 0, \quad (5)$$

where the semicolon denotes the covariant derivative. This gauge implies that $h_{b,a}^a = \frac{1}{2}h_{,b} - \bar{\Gamma}_{ca}^a h_b^c - \bar{\Gamma}_{ba}^c h_c^a$, $\bar{\Gamma}_{bc}^a$, being the second kind Christoffel symbols of the 5D background metric. If we consider an apparent vacuum $\bar{\mathcal{R}}_{ab} = 0$, we obtain

$$\square h_{ab} = \bar{g}^{ce} h_{ce,ab} - (h_{a,cb}^c + h_{b,ac}^c), \quad (6)$$

so that the equation (6) becomes

$$\square h_{ab} = (\bar{\Gamma}_{ec}^c h_a^e)_{,b} + (\bar{\Gamma}_{ea}^c h_b^e)_{,c} - (\bar{\Gamma}_{ac}^e h_e^c)_{,b} - (\bar{\Gamma}_{ba}^e h_e^c)_{,c}, \quad (7)$$

which is the linearized wave equation for the tensor field h_{ab} on any 5D Ricci-flat metric, and

$$\square \equiv \bar{g}^{\alpha\beta} \frac{\partial^2}{\partial x^\alpha \partial x^\beta} + 2\bar{g}^{\alpha\psi} \frac{\partial^2}{\partial x^\alpha \partial \psi} + \bar{g}^{\psi\psi} \frac{\partial^2}{\partial \psi^2}.$$

The equation (7) describes the dynamics of $h_{ab}(x^a)$ on the 5D Ricci-flat canonical metric (2). Notice that they have the form $\square h_{ab} = \mathcal{S}_{ab}$. Here, \mathcal{S}_{ab} describes the interaction of h_{ab} with the background metric, which manifests itself through the connections $\bar{\Gamma}_{ac}^e$. Of course \mathcal{S}_{ab} should be null for a really 5D Riemann-flat metric [i.e., for a 5D Minkowsky spacetime].

B. Fluctuations on a 5D apparent vacuum

In order to study the tensor fluctuations h_{ab} , on the canonical metric (2), we shall consider the Einstein equations $G_{ab} = -8\pi G T_{ab}$, but written in the following manner

$$R_{ab} = -8\pi G \left[T_{ab} - \frac{1}{3}g_{ab}T \right], \quad (8)$$

where we have considered the expressions $g^{ab}g_{ab} = 5$ with $\bar{g}^{ab}\bar{g}_{ab} = 5^3$, $\mathcal{R}_{ab} = \bar{\mathcal{R}}_{ab} + \delta\mathcal{R}_{ab}$, $T_{ab} = \bar{T}_{ab} + \delta T_{ab}$, and the expansion $g_{ab} = \bar{g}_{ab} + h_{ab}$ for the 5D Ricci-flat metric. If we

³ Both expressions imply that $h^2 + \bar{g}^{ab}h_{ab} + h^{ab}\bar{g}_{ab} = 0$.

separate the equations on the background and fluctuations in (8), one obtains

$$\bar{\mathcal{R}}_{ab} = -8\pi G \left[\bar{T}_{ab} - \frac{1}{3} \bar{g}_{ab} \bar{T} \right], \quad (9)$$

$$\delta \mathcal{R}_{ab} = -8\pi G \left[\delta T_{ab} - \frac{1}{3} (\bar{g}_{ab} + h_{ab}) \delta T \right]. \quad (10)$$

The equation (9) describes the background Einstein equations on the 5D vacuum and (10) the tensor fluctuations around (2). By multiplying the equation (9) by \bar{g}^{ab} and taking into account that the metric is Ricci-flat, we obtain

$$\bar{T} = 0, \quad (11)$$

which is a manifestation of the absence of matter on 5D. On the other hand, from the equation (10), it is possible to obtain the expression that relates the fluctuations of the Ricci scalar and the energy-momentum one

$$\delta \mathcal{R} = \frac{16\pi G}{3} \delta T. \quad (12)$$

Now we consider the expression (4), which is *valid only in a linear approximation*. Hence, the equation (6) holds

$$\delta T_{ab} = \frac{1}{3} (\bar{g}_{ab} + h_{ab}) \delta T, \quad \text{or} \quad \delta T_{ab} = \frac{1}{3} g_{ab} \delta T. \quad (13)$$

Therefore, from the equation (10) one obtains

$$\delta \mathcal{R}_{ab} = 0, \quad \delta T_{ab} = 0, \quad (14)$$

so that, as one expects, \mathcal{R} and T are 5D invariants.

III. DYNAMICS OF GW ON 4D

Now we consider the dynamics of h_{ab} on a 4D hypersurface described by $\bar{g}_{\alpha\beta}(x^\alpha, \psi)$ evaluated on a particular foliation on the noncompact fifth coordinate $\psi = \psi_0$:

$$\bar{g}_{\alpha\beta}(x^\alpha, \psi)|_{\psi=\psi_0}. \quad (15)$$

We shall consider that the components $h_{\psi\psi} = h_{a\psi} = h_{\psi a} = 0$. The equation (6), evaluated on this hypersurface can be written as

$$\begin{aligned} {}^{(4)}\square h_{ab} + 2\bar{g}^{\alpha\psi} \frac{\partial^2 h_{ab}}{\partial x^\alpha \partial \psi} + \bar{g}^{\psi\psi} \frac{\partial^2 h_{ab}}{\partial \psi^2} \Big|_{\psi=\psi_0} &= \frac{\partial}{\partial \psi} (\bar{\Gamma}_{\alpha\alpha}^\psi h_b^\alpha) + \frac{\partial}{\partial x^\beta} (\bar{\Gamma}_{\alpha\alpha}^\beta h_b^\alpha - \bar{\Gamma}_{ba}^\alpha h_\alpha^\beta) \\ &+ \frac{\partial}{\partial x^\beta} (\bar{\Gamma}_{\beta\alpha}^\alpha h_a^\beta + \bar{\Gamma}_{\beta\psi}^\psi h_a^\beta - \bar{\Gamma}_{a\beta}^\alpha h_\alpha^\beta) \Big|_{\psi=\psi_0}, \end{aligned} \quad (16)$$

where we must remember that the connections $\bar{\Gamma}_{bc}^a$ are those of the 5D canonical metric (2). Since we are interested on the $\mu\nu$ components of h_{ab} , we obtain

$$\begin{aligned} {}^{(4)}\square h_{\mu\nu} + 2\bar{g}^{\alpha\psi} \frac{\partial^2 h_{\mu\nu}}{\partial x^\alpha \partial \psi} + \bar{g}^{\psi\psi} \frac{\partial^2 h_{\mu\nu}}{\partial \psi^2} \Big|_{\psi=\psi_0} &= \frac{\partial}{\partial \psi} (\bar{\Gamma}_{\alpha\mu}^\psi h_\nu^\alpha) + \frac{\partial}{\partial x^\beta} (\bar{\Gamma}_{\alpha\mu}^\beta h_\nu^\alpha - \bar{\Gamma}_{\nu\mu}^\alpha h_\alpha^\beta) \\ &+ \frac{\partial}{\partial x^\nu} (\bar{\Gamma}_{\beta\alpha}^\alpha h_\mu^\beta + \bar{\Gamma}_{\beta\psi}^\psi h_\mu^\beta - \bar{\Gamma}_{\mu\beta}^\alpha h_\alpha^\beta) \Big|_{\psi=\psi_0}. \end{aligned} \quad (17)$$

The equations of motion for the remaining components are complied automatically because $h_{\psi\psi} = h_{\alpha\psi} = h_{\psi a} = 0$. If we take into account that $\bar{g}^{\alpha\psi} = 0$ and $\bar{g}^{\psi\psi} = -1$, we obtain

$$\begin{aligned} {}^{(4)}\square h_{\mu\nu} - \frac{\partial^2 h_{\mu\nu}}{\partial \psi^2} \Big|_{\psi=\psi_0} &= \frac{\partial}{\partial \psi} (\bar{\Gamma}_{\alpha\mu}^\psi h_\nu^\alpha) + \frac{\partial}{\partial x^\beta} (\bar{\Gamma}_{\alpha\mu}^\beta h_\nu^\alpha - \bar{\Gamma}_{\nu\mu}^\alpha h_\alpha^\beta) \\ &+ \frac{\partial}{\partial x^\nu} (\bar{\Gamma}_{\beta\alpha}^\alpha h_\mu^\beta + \bar{\Gamma}_{\beta\psi}^\psi h_\mu^\beta - \bar{\Gamma}_{\mu\beta}^\alpha h_\alpha^\beta) \Big|_{\psi=\psi_0}. \end{aligned} \quad (18)$$

Since we are dealing with 5D canonical metrics like (2), it is possible to consider the following separation of variables: $h_{\mu\nu}(x^\alpha, \psi) \sim \Sigma(\psi) \tilde{h}_{\mu\nu}(x^\alpha)$. We obtain the system of equations

$${}^{(4)}\square \tilde{h}_{\mu\nu} - m_g^2 \tilde{h}_{\mu\nu} \Big|_{\psi=\psi_0} = \mathcal{S}_{\mu\nu}(x^\alpha, \psi_0), \quad (19)$$

$$\frac{\partial^2 \Sigma(\psi)}{\partial \psi^2} = m_g^2 \Sigma(\psi), \quad (20)$$

where

$$\mathcal{S}_{\mu\nu}(x^\alpha) = \frac{\partial}{\partial \psi} (\bar{\Gamma}_{\alpha\mu}^\psi \tilde{h}_\nu^\alpha) + \frac{\partial}{\partial x^\beta} (\bar{\Gamma}_{\alpha\mu}^\beta \tilde{h}_\nu^\alpha - \bar{\Gamma}_{\nu\mu}^\alpha \tilde{h}_\alpha^\beta) + \frac{\partial}{\partial x^\nu} (\bar{\Gamma}_{\beta\alpha}^\alpha \tilde{h}_\mu^\beta + \bar{\Gamma}_{\beta\psi}^\psi \tilde{h}_\mu^\beta - \bar{\Gamma}_{\mu\beta}^\alpha \tilde{h}_\alpha^\beta) \Big|_{\psi=\psi_0}. \quad (21)$$

The equation (19) is the Fierz-Pauli[12] equation of motion for massive gravitons with mass m_g and a $\mathcal{S}_{\mu\nu}$ -source. Notice that m_g is induced by the foliation $\psi = \psi_0$, but the source becomes from the nonzero connections $\bar{\Gamma}_{\beta\gamma}^\alpha$ of the background canonical metric (2). It is evident from the equation (21) that the source $\mathcal{S}_{\mu\nu}$ is originated in the interaction of the gravitational waves with the background through the connections $\bar{\Gamma}_{\beta\gamma}^\alpha$ of the 5D background metric (2) evaluated in the static foliation $\psi = \psi_0$. The general solution of (19) is

$$\begin{aligned} \tilde{h}_{\mu\nu}(x) &= \int d^4 x' \Delta(x - x') \mathcal{S}_{\mu\nu}(x') \\ &= \int d^4 x' \Delta(x - x') \left\{ \frac{\partial}{\partial x'^\beta} (\bar{\Gamma}_{\alpha\mu}^\beta \tilde{h}_\nu^\alpha) - \frac{\partial}{\partial x'^\beta} (\bar{\Gamma}_{\nu\mu}^\alpha \tilde{h}_\alpha^\beta) \right. \\ &\quad \left. + \frac{\partial}{\partial x'^\nu} (\bar{\Gamma}_{\beta\alpha}^\alpha \tilde{h}_\mu^\beta) - \frac{\partial}{\partial x'^\nu} (\bar{\Gamma}_{\mu\beta}^\alpha \tilde{h}_\alpha^\beta) \right\}, \end{aligned} \quad (22)$$

where $\Delta(x - x')$ is the Green function which obeys: $((^{(4)}\square - m_g^2) \Delta(x - x') = \delta^{(4)}(x - x')$. The effective action due to the interaction of GW with matter will be

$$\mathcal{I}_{Int} = 4\pi G \int d^4x \bar{T}^{\mu\nu}(x) \tilde{h}_{\mu\nu}(x), \quad (23)$$

where $\tilde{h}_{\mu\nu}(x)$ is given by (22) and the source $\mathcal{S}_{\mu\nu}$ by (21).

IV. CONCLUDING REMARKS

We have studied GW in the framework of the Induced Matter theory of gravity. In particular, we have restricted our study to canonical metrics like (2), which are, at least, Ricci-flat (they could be also Riemann-flat). These kind of metrics are suitable to describe a 5D physical vacuum on which GW propagates freely without interactions. We have defined a 5D gauge invariant field Ψ_b^a , to obtain the linearized equations (7) that describe gravitational waves (for massless gravitons) on a canonical Ricci-flat metric (2). Note that they take the form $\square h_{ab} = \mathcal{S}_{ab}$, where \mathcal{S}_{ab} is the interaction of the GW with the background metric, which should be null for a really 5D Riemann-flat metric [i.e., for a 5D Minkowsky spacetime]. After it, we have obtained the equations of motion for $h_{\mu\nu}$, on a static foliation of the metric (2). From the relativistic point of view, observers that move with frames $U^4 \equiv \frac{dx^4}{dS} = 0$ (described by a constant foliation on the extra dimension), can see the massive gravitons moving on a curved hypersurface, such that their equation of motion is described by the effective 4D Einstein's equations (19), with a non-zero fluctuations of the energy-momentum tensor: $\mathcal{S}_{\mu\nu}$ given by (21). Such a 4D hypersurface is embedded in the 5D apparent vacuum, which is geometrically described by a 5D Ricci-flat spacetime. From the mathematical point of view, the Campbell-Magaard theorem[11] serves as a ladder to go between manifolds whose dimensionality differs by one. This theorem, which is valid in any number of dimensions, implies that every solution of the 4D Einstein equations with arbitrary energy-momentum tensor can be embedded, at least locally, in a solution of the 5D Einstein field equations in vacuum.

The interesting of the equation (19) is that describes gravitons with mass m_g on a 4D curved spacetime, where $\mathcal{S}_{\mu\nu}$ represents the interaction of GW with the connections of the background 5D metric (2) on $\psi = \psi_0$. This is the main difference with the original Fierz-Pauli formalism, where $\mathcal{S}_{\mu\nu}$ becomes from the interaction of GW with matter described by

$\bar{T}_{\mu\nu}$ [3].

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